Lecture 37

HWK Comment:

- Given $A \in M_{n \times n}$ and $\vec{x} \in \mathbb{R}^n$ $\exists$ two approaches to finding $A^k \vec{x}$. Let $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ be a basis of e-vecs for A. Then

1) Diagonalize $A = S \Lambda S^{-1}$, $S = [\vec{v}_1, \dots, \vec{v}_n]$

$$A^k \vec{x} = S \Lambda^k S^{-1} \vec{x}, \quad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

2) $\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$

$$A^k \vec{x} = c_1 A^k \vec{v}_1 + \dots + c_n A^k \vec{v}_n$$

$$= c_1 \lambda_1^k \vec{v}_1 + \dots + c_n \lambda_n^k \vec{v}_n$$

$\boxed{\text{E.g.}}$ $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$ Diagonilize A

e-vals	7	-1
e-vecs	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$S = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \qquad A = S \Lambda S^{-1}$$

$$\Lambda = \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}$$

- Since S is $\perp$-matrix we know $S^T S = I$
  ↳ so $S^{-1} = S^T$

So in this case $A = S \Lambda S^T$

Spectral Theorem: A matrix $A \in M_{n\times n}$ has an ortho-normal basis of e-vecs $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ i.f.f. $A = A^T$ is symmetric.

Rmk: In particular A has n real e-vals (counting with multiplicity)

Ex quadratic terms

$$\boxed{ax^2 + bxy + cy^2} + dx + ey + f = 0$$

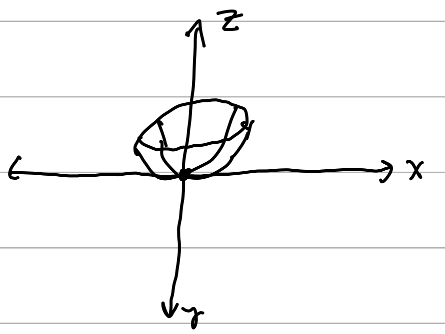
$$A = \begin{bmatrix} 0 & b/2 \\ b/2 & c \end{bmatrix} \leftarrow \text{matrix for quadratic terms}$$

fact:

conic section	ellipse	hyperbola	parabola	line or less
evals of A	same sign	opposite sign	one e-val is 0	both e-vals are 0

Ex $f : \mathbb{R}^2 \to \mathbb{R}^3$

$f(x,y) = z$



critical points: $\frac{\partial f}{dx} = \frac{\partial f}{dy} = 0$

Second derivative test:
$$\begin{bmatrix} \frac{\partial^2 f}{dx\,dx} & \frac{\partial^2 f}{dx\,dy} \\ \frac{\partial^2 f}{\partial y\,\partial x} & \frac{\partial^2 f}{\partial y\,\partial y} \end{bmatrix}$$

Symmetric matrix

(Jacobian I think)

Singular Value Decomposition

- $A \in M_{m \times n}$ (Assume $m \geq n$)

<u>Obs</u>: $A^TA \in M_{n \times n}$ is Symmetric

and all e-vals are ≥ 0.

- Let $\sigma_1, \ldots, \sigma_n$ be $\sqrt{\text{e-vals}}$. These are called <u>singular values</u> of A.

Singular Value Decomposition Theorem:

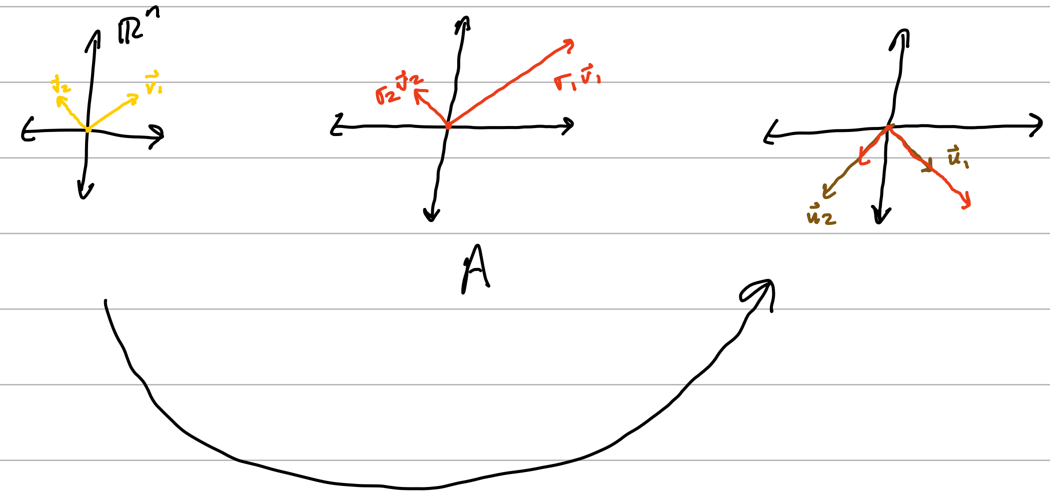
- $\forall$ $A \in M_{n \times n}$ and $m \geq n$ $\exists$ $\perp$-normal sets $\{\vec{u}_1, \ldots, \vec{u}_n\} \subset \mathbb{R}^m$, $\{\vec{v}_1, \ldots, \vec{v}_n\} \subset \mathbb{R}^n$

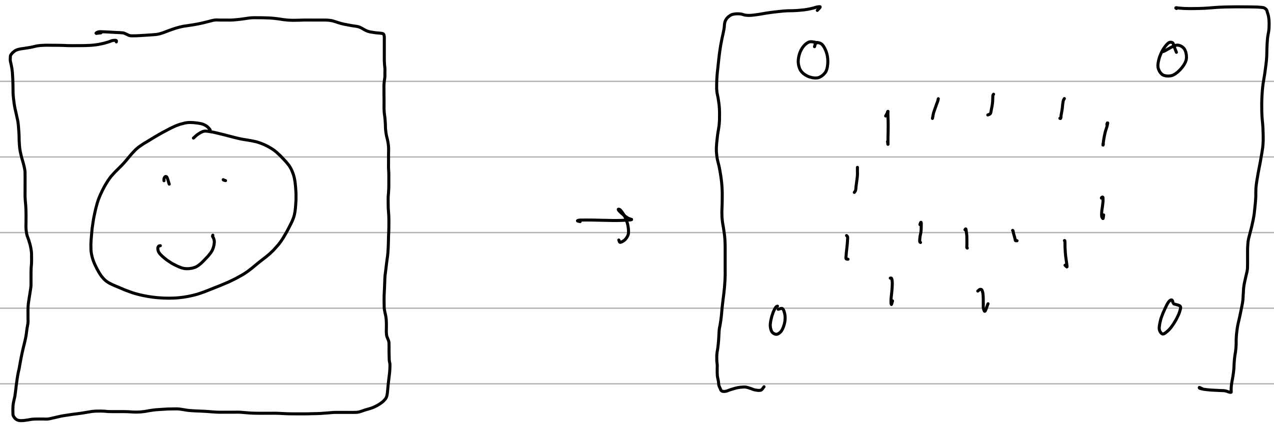
such that

$$A = U \Sigma V^T$$

where $U = [\vec{u}_1 \ldots \vec{u}_n]$, $V = [\vec{v}_1 \ldots \vec{v}_n]$, $\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$

$\boxed{\text{Eg}}$





Matrix where if pixel is on in the the image, then the entry is a 1, if pixel is off then entry is 0.